

# A FREE PRODUCT PAIR RIGIDITY RESULT IN VON NEUMANN ALGEBRAS

YOSHIMICHI UEDA

ABSTRACT. We prove that the free product pair of any finitely many copies of the unique amenable type III<sub>1</sub> factor endowed with weakly mixing states remembers the number of free components and the given states.

## 1. INTRODUCTION

The free product construction in von Neumann algebras is a way to construct a pair of von Neumann algebra and faithful normal state from given such pairs. Here we are interested in how much information about the given pairs can be recovered from the resulting pair. Moreover, we are particularly interested in analysis of free products whose free component algebras as well as centralizers are all amenable, because the available technologies showing rigidity have worked a little for such free products due to the lack of suitable non-amenable von Neumann subalgebras in appearance (see [13]).

Let  $R_\infty$  be the unique amenable type III<sub>1</sub> factor and  $\{\varphi_i\}_{i=1}^m$  and  $\{\psi_j\}_{j=1}^n$ , be finite families of weakly mixing states on  $R_\infty$ . (See the beginning of section 3 for the definition of weakly mixing states.) Remark here that a von Neumann algebra with weakly mixing states must be either the trivial algebra or a type III<sub>1</sub> factor. (This simple fact has been well known in the algebraic quantum field theory, see e.g., [3, Corollary 1.0.8].) Consider two free product pairs  $(M, \varphi) := \star_{i=1}^m (R_\infty, \varphi_i)$  and  $(N, \psi) := \star_{j=1}^n (R_\infty, \psi_j)$ . The resulting  $M$  and  $N$  are known to be type III<sub>1</sub> factors and furthermore the centralizers  $M_\varphi$  and  $N_\psi$  are also known to be trivial ([2, Lemma 7]; see also the proof of [25, Proposition 2.1]). In what follows,  $(M, \varphi) \cong (N, \psi)$  means that there exists a bijective  $*$ -homomorphism  $\pi : M \rightarrow N$  such that  $\psi = \pi_*(\varphi) := \varphi \circ \pi^{-1}$ . We will explicitly write the canonical embedding maps  $\lambda_i^M : R_\infty \rightarrow M$  ( $1 \leq i \leq m$ ) and  $\lambda_j^N : R_\infty \rightarrow N$  ( $1 \leq j \leq n$ ) so that  $\varphi \circ \lambda_i^M = \varphi_i$  and  $\psi \circ \lambda_j^N = \psi_j$ . The main theorem of this paper is the next new rigidity phenomenon, which we call the *free product pair rigidity*.

**Theorem 1.** *If there exists a bijective  $*$ -homomorphism  $\pi : M \rightarrow N$  with  $\psi = \pi_*(\varphi)$ , then  $m = n$  and there exists a permutation  $\kappa \in \mathfrak{S}_m$  such that for each  $1 \leq i \leq m$ ,  $\pi(\lambda_i^M(R_\infty)) = \lambda_{\kappa(i)}^N(R_\infty)$  holds and  $\pi_i := (\lambda_{\kappa(i)}^N)^{-1} \circ (\pi \upharpoonright_{\lambda_i^M(R_\infty)}) \circ \lambda_i^M \in \text{Aut}(R_\infty)$  satisfies  $\psi_{\kappa(i)} = (\pi_i)_*(\varphi_i)$ .*

The theorem says that  $(M, \varphi) \cong (N, \psi)$  forces that  $m = n$  and  $(R_\infty, \varphi_i) \cong (R_\infty, \psi_i)$ ,  $1 \leq i \leq m$ , after permutation on the indices. To the best of the author's knowledge this kind of complete restoration has not been known so far. The next corollary is immediate from the above theorem.

---

*Date:* Oct. 1st, 2016.

*2010 Mathematics Subject Classification.* 46L10, 46L54, 46L36.

*Key words and phrases.* Free product; Type III factor; Intertwining technique; Weakly mixing action; Central sequence.

**Corollary 2.** *If all the  $\varphi_i$  are identical to a fixed weakly mixing state  $\varphi_0$ , then the  $\varphi$ -preserving automorphism group  $\text{Aut}_\varphi(M)$  is isomorphic to the wreath product group  $\text{Aut}_{\varphi_0}(R_\infty) \wr \mathfrak{S}_m = \text{Aut}_{\varphi_0}(R_\infty)^m \rtimes \mathfrak{S}_m$ .*

Although our main targets are free products of amenable von Neumann algebras, we would like to point out that *the above statements still hold even if some copies of  $R_\infty$  in the initial and/or the target free products are replaced with anti-freely indecomposable non-amenable type  $\text{III}_1$  factors (in the sense of [13]) endowed with weakly mixing states*. The proof is done only by replacing Lemma 7 below with the intermediate assertion ( $\diamond$ ) in the proof of [13, Main Theorem].

The above theorem can be regarded as a kind of Kurosh-type result, and we remark that, unlike the above theorem, all the previous Kurosh-type results require the non-amenability plus ‘structure’ of component algebras; see [13] and all related references therein such as Ozawa’s pioneer work in the direction. Here we do not assume such an assumption. Instead, we use a simple analysis of central sequences in the presence of weak mixing property. The required ‘rigidity’ comes from the given states (or more precisely their modular actions) and the required ‘malleability’ is nothing but the amenability of component algebras. The same technique can basically be applied to any free group factors of finite rank, though it is unclear at the moment of this writing whether or not the technique is useful even for their (non-)isomorphism problem (see the concluding remarks).

Necessary backgrounds can be found in [24, section 2], and the other necessary facts will be referred to suitable references at appropriate places. Unlike [24, section 2] we will use the standard form  $(M, L^2(M), J^M, \mathfrak{P}^M)$  of a given von Neumann algebra  $M$  (see e.g., [22, Chapter IX, section 1]) instead of the GNS representation associated with a specific faithful normal state, and it is well known that any positive  $\varphi \in M_*$  has a unique representing vector  $\xi_\varphi \in \mathfrak{P}$ , i.e.,  $(x\xi_\varphi | \xi_\varphi)_{L^2(M)} = \varphi(x)$  for  $x \in M$  and moreover that  $\xi_\varphi$  becomes cyclic and separating if and only if  $\varphi$  is faithful. We also use the notation concerning ultraproduct von Neumann algebras such as  $(x_n)^\omega$  in [1], which is a bit different from that in [24, section 2]. In what follows, we say that a von Neumann subalgebra is *with expectation*, if there exists a faithful normal conditional expectation from the ambient von Neumann algebra onto it.

## 2. YET ANOTHER VARIANT OF POPA’S CRITERION

Let  $M$  be a  $\sigma$ -finite von Neumann algebra and  $A, B \subseteq M$  be two (not necessarily unital) von Neumann subalgebras with a faithful normal conditional expectation  $E_B : 1_B M 1_B \rightarrow B$ . The fact below was observed in fall, 2014 and triggered by a discussion with Yusuke Isono about an unpublished attempt due to S. Deprez and S. Raum, which also triggered Houdayer and Isono’s thoroughgoing work [10, section 4] on Popa’s criterion (see [18, Appendix], [19, section 2]). Only the new input here is the use of amenability in place of the so-called minimal distance theorem in Hilbert spaces so that the result below requires  $A$  to be amenable, though it is not necessary either  $A$  or  $B$  is finite. The proof is so simple, and moreover the consequence is necessary later and enough in many actual applications; hence we sketch the proof with emphasis on the main points.

**Proposition 3.** *Assume that  $A$  is amenable and has separable predual. Then the following are equivalent:*

- (1) *There exists no net  $u_i$  of unitaries in  $A$  satisfies that  $\lim_i E_B(y^* u_i x) = 0$  strongly for all  $x, y \in 1_A M 1_B$ .*

- (2) *There exist a natural number  $n$ , a non-zero partial isometry  $v \in \mathbb{M}_{n,1}(M)$  and a normal (possibly non-unital)  $*$ -homomorphism  $\theta : A \rightarrow \mathbb{M}_n(B)$  such that  $va = \theta(a)v$  for every  $a \in A$ .*

Following Popa's notation, we write  $A \preceq_M B$  and say that  $A$  embeds  $B$  inside  $M$ , if the above equivalent conditions hold.

*Proof.* It suffices to show that item (1) implies item (2), for the opposite direction is shown in the usual way (see [27, Proposition C.1 (1)  $\Rightarrow$  (4)]).

Let  $z$  be the maximal central projection in  $B$  so that  $Bz$  is finite; hence  $B(1_B - z)$  is properly infinite. Choose and fix a faithful normal state  $\varphi$  on  $B$  in such a way that  $\varphi|_{Bz}$  is tracial. Remark that item (1) is equivalent to that there exist a finite subset  $\mathcal{F} \in 1_A M 1_B$  and  $\varepsilon > 0$  so that  $\sum_{x,y \in \mathcal{F}} \|E_B(y^*ux)\xi_\varphi\|_{L^2(N)}^2 \geq \varepsilon$  for all unitaries  $u \in A$ .

Consider the unitalization  $B^\sim := B + \mathbb{C}1_B^\perp$  inside  $M$ , and we can extend  $E_B$  and  $\varphi$  to a faithful normal conditional expectation  $(E_B)^\sim : M \rightarrow B^\sim$  and a faithful normal positive linear functional  $\varphi^\sim$  on  $B^\sim$ , respectively.

Applying the well-known representation theory of von Neumann algebras to the restriction of the right action of  $M$  on  $L^2(M)$  to  $B^\sim$ , we can identify the basic extension  $\langle M, B^\sim \rangle$  (by  $(E_B)^\sim$ ) with a certain reduced algebra  $P(B(\ell^2) \bar{\otimes} B^\sim)P$  with projection  $P \in B(\ell^2) \bar{\otimes} B^\sim$  in such a way that  $e_{11} \otimes 1 \leq P$  and that the identification sends  ${}^t r := J^M r J^M$  with projection  $r \in \mathcal{Z}(B^\sim)$ , the Jones projection  $e_{B^\sim}$  and  $b^\sim e_{B^\sim}$  with  $b^\sim \in B^\sim$  to  $P(1_{B(\ell^2)} \otimes r) = (1_{B(\ell^2)} \otimes r)P$ ,  $e_{11} \otimes 1$  and  $e_{11} \otimes b^\sim$ , respectively, where  $e_{ij}$  are canonical matrix units in  $B(\ell^2)$ . See the proof of [26, Proposition 3.1 (i)  $\Rightarrow$  (ii)]. In particular, we have  ${}^t r e_{B^\sim} = r e_{B^\sim}$  ( $= e_{11} \otimes r$  in  $P(B(\ell^2) \bar{\otimes} B^\sim)P$ ). Hence we can construct a faithful normal semifinite trace  $\text{Tr}$  on  ${}^t z \langle M, B^\sim \rangle$  by transferring  $\text{Tr}_{B(\ell^2)} \bar{\otimes} (\varphi|_{Bz})$  on  $B(\ell^2) \bar{\otimes} Bz$  to  ${}^t z \langle M, B^\sim \rangle$ . Remark that  $\text{Tr}$  is characterized by  $\text{Tr}(b e_{B^\sim}) = \varphi(b)$  for  $b \in Bz$  because  $b e_{B^\sim} = b z e_{B^\sim} = b {}^t z e_{B^\sim} \in {}^t z \langle M, B^\sim \rangle$ , which corresponds to  $e_{11} \otimes b$  in  $B(\ell^2) \bar{\otimes} Bz$ .

Let  $\widehat{(E_B)^\sim} : \langle M, B \rangle \rightarrow M$  be the dual operator valued weight (see e.g., [15, subsection 2.1]). Set  $d := \sum_{y \in \mathcal{F}} y e_{B^\sim} y^* \in 1_A {}^t 1_B \langle M, B^\sim \rangle 1_A {}^t 1_B$ , and then  $\widehat{(E_B)^\sim}(d) = \sum_{y \in \mathcal{F}} y y^*$ . We have

$$\text{Tr}({}^t z d) = \sum_{y \in \mathcal{F}} \text{Tr}({}^t z y e_{B^\sim} ({}^t z y e_{B^\sim})^*) = \sum_{y \in \mathcal{F}} \text{Tr}({}^t z e_{B^\sim} y^* y e_{B^\sim}) = \sum_{y \in \mathcal{F}} \varphi(z E_B(y^* y)) < +\infty.$$

Set  $\mathcal{C} := \overline{\text{co}}^{\sigma\text{-w}}\{u^* du \mid \text{unitary } u \in A\}$  inside  $1_A {}^t 1_B \langle M, B^\sim \rangle 1_A {}^t 1_B$ . Since  $\mathcal{C}$  is a  $\sigma$ -weakly compact convex subset and since  $A$  is amenable with separable predual, there exists a fixed point  $c_0 \in \mathcal{C}$  under the adjoint action of the unitary group of  $A$ . (See the proof of [15, Theorem 3.9] if the reader needs the details.) Using the equivalent condition of item (1) given in the second paragraph, we have

$$\sum_{x \in \mathcal{F}} ((u^* du) x \xi_{\varphi^\sim \circ (E_B)^\sim} \mid x \xi_{\varphi^\sim \circ (E_B)^\sim})_{L^2(M)} = \sum_{x,y \in \mathcal{F}} \|E_B(y^*ux)\xi_\varphi\|_{L^2(N)}^2 \geq \varepsilon$$

for every unitary  $u \in A$ , and hence  $\sum_{x \in \mathcal{F}} (c x \xi_{\varphi^\sim \circ (E_B)^\sim} \mid x \xi_{\varphi^\sim \circ (E_B)^\sim})_{L^2(M)} \geq \varepsilon$  for all  $c \in \mathcal{C}$ . In particular,  $c_0$  is non-zero. Since  $\widehat{(E_B)^\sim}(d) \in M$ , we have  $\chi \circ \widehat{(E_B)^\sim}(c_0) \leq \|\widehat{(E_B)^\sim}(d)\| \chi$  for every positive  $\chi \in M_*$  by lower semicontinuity, and thus it is plain to see, by using its generalized spectral decomposition (see [7, Theorem 1.5]), that  $\widehat{(E_B)^\sim}(c_0)$  falls in  $M$ . Similarly, by lower semicontinuity we have  $\text{Tr}({}^t z c_0) \leq \text{Tr}({}^t z d) < +\infty$ . Taking a suitable spectral projection of  $c_0$  we can find a non-zero projection  $f \in A' \cap 1_A {}^t 1_B \langle M, B^\sim \rangle 1_A {}^t 1_B$  in such a way that  $\widehat{(E_B)^\sim}(f) \in M$  and  $\text{Tr}(p) < \infty$  with  $p := {}^t z f \in {}^t z \langle M, B^\sim \rangle$ . With  $\langle M, B^\sim \rangle = P(B(\ell^2) \bar{\otimes} B^\sim)P$ , the projection  ${}^t(1_B - z)e_B$  is nothing but  $e_{11} \otimes (1_B - z)$  being properly infinite, since  $B(1_B - z)$  is properly

infinite; hence  $q := {}^t(1_B - z)f = f - p$  is subequivalent to  ${}^t(1_B - z)e_{B^\sim}$  inside  $\langle M, B^\sim \rangle$  by [16, Theorem 6.3.4]. By the same reason as in the beginning of the proof of [4, Proposition F.10] or by [27, Lemma A.1] we may and do assume, with cutting  $p$  by a central projection if necessary, that  $p$  is subequivalent to  $\text{diag}({}^tze_{B^\sim}, \dots, {}^tze_{B^\sim})$  inside the  $n$ -amplification  $\mathbb{M}_n({}^tze_{B^\sim})$  for some finite  $n$ . Consequently, there exists a partial isometry  $V \in \mathbb{M}_n(\langle M, B^\sim \rangle)$  so that  $V^*V = \text{diag}(f, 0, \dots, 0)$  and  $VV^* \leq e_B^{(n)} := \text{diag}(e_B, \dots, e_B)$  with  $e_B := {}^t1_B e_{B^\sim} = 1_B e_{B^\sim}$  as before. Define a normal  $*$ -homomorphism  $\theta : A \rightarrow \mathbb{M}_n(B)$  is defined by

$$a \in A \mapsto V \text{diag}(af, 0, \dots, 0) V^* \in e_B^{(n)} \mathbb{M}_n(\langle M, B^\sim \rangle) e_B^{(n)} = \mathbb{M}_n(B) e_B^{(n)} \cong \mathbb{M}_n(B).$$

It follows that  $V \text{diag}(a, 0, \dots, 0) = \theta(a)V$  for every  $a \in A$ . Apply  $\widehat{(E_B)^\sim} \otimes \text{Id}_n$  to this equation (*n.b.*, we can do since  $\widehat{(E_B)^\sim}(f) \in M$ ), and the push-down lemma [15, Proposition 2.2] (which clearly holds without the factoriality assumption on given algebras) with the help of (the proof of) [7, Lemma 4.5] shows that there exists  $w \in \mathbb{M}_n(M)$  such that  $w^*w \leq \text{diag}(1, 0, \dots, 0)$  and  $w \text{diag}(a, 0, \dots, 0) = \theta(a)w$  for every  $a \in A$ . Taking the polar decomposition of  $w$  we get the desired element  $v$ .  $\square$

**Remark 4.** Proposition 3 is a bit strengthened, as Houdayer and Isono [10] did in the following way: If one further assumes that there exists a faithful normal conditional expectation  $E_A : 1_A M 1_A \rightarrow A$ , then one can make the inclusion  $\theta(A) \subseteq \theta(1_A) \mathbb{M}_n(B) \theta(1_A)$  with expectation.

*Proof.* Let  $f$  be as in the above proof; namely,  $f$  is a non-zero projection in  $1_A {}^t1_B \langle M, B^\sim \rangle 1_A {}^t1_B$  such that  $\text{diag}(f, 0 \dots 0) \precsim e_B^{(n)}$  inside  $\mathbb{M}_n(\langle M, B^\sim \rangle)$  and  $\widehat{(E_B)^\sim}(f) \in M$ . Set  $\Phi := (E_A)^\sim \circ \widehat{(E_B)^\sim}$ , a faithful normal operator valued weight from  $\langle M, B^\sim \rangle$  onto  $A^\sim$ , where  $A^\sim$  is the unitalization of  $A$  inside  $M$  and  $(E_A)^\sim : M \rightarrow A^\sim$  is a faithful normal conditional expectation extending  $E_A$ . Observe  $\Phi(f) \in \mathcal{Z}(A)$ , and thus one can choose a non-zero spectral projection  $e \in \mathcal{Z}(A)$  of  $\Phi(f)$  with  $\Phi(f)e \geq \delta e$  for some  $\delta > 0$ . Observe that  $ef$  is still a non-zero projection in  $A' \cap 1_A {}^t1_B \langle M, B^\sim \rangle 1_A {}^t1_B$  (since  $e \in \mathcal{Z}(A^\sim)$ ). Replacing  $f$  with  $ef$  we may and do assume that  $\Phi(f) \geq \delta s$  and  $f \leq s$  with denoting by  $s$  the support projection of  $\Phi(f)$ . We can choose a non-zero positive element  $c \in \mathcal{Z}(A)$  so that  $\Phi(f)c = c\Phi(f) = s$ . The mapping  $x \in f \langle M, B^\sim \rangle f \mapsto \Phi(x)cf \in Af$  gives a faithful normal conditional expectation. Therefore, the desired assertion follows, since  $Af \subseteq f \langle M, B^\sim \rangle f$  is conjugate to  $\theta(A) \subseteq \theta(1_A) \mathbb{M}_n(B^\sim) \theta(1_A)$  by the composition of the mapping

$$x \in f \langle M, B^\sim \rangle f \mapsto \text{diag}(x, 0 \dots 0) \mapsto V \text{diag}(x, 0 \dots 0) V^* \in e_B^{(n)} \mathbb{M}_n(\langle M, B^\sim \rangle) e_B^{(n)}$$

and the inverse of  $y \in \mathbb{M}_n(B) \mapsto y e_B^{(n)} \in \mathbb{M}_n(B) e_B^{(n)} = e_B^{(n)} \mathbb{M}_n(\langle M, B^\sim \rangle) e_B^{(n)}$  in this order.  $\square$

### 3. PROOF OF THEOREM 1

We begin by recalling the weak mixing property for group actions as well as for faithful normal states. Let  $M$  be a von Neumann algebra with a distinguished faithful normal state  $\varphi$ . A  $\varphi$ -preserving action  $\alpha : G \curvearrowright N$  of a second countable, locally compact group  $G$  is said to be weakly mixing if its canonical implementing unitary representation of  $\pi : G \curvearrowright L^2(M)$  (defined by  $\pi(g)x\xi_\varphi := \alpha_g(x)\xi_\varphi$ ,  $x \in M$ ) enjoys that for every finite subset  $\Omega$  of a given dense subset of  $L^2(M) \ominus \mathbb{C}\xi_\varphi$  and every  $\varepsilon > 0$  there exists a  $g \in G$  such that  $|(\pi(g)\xi|\zeta)_{L^2(M)}| < \varepsilon$  for all  $\xi, \zeta \in \Omega$ . We also say that the state  $\varphi$  itself is weakly mixing if its modular automorphism group  $\sigma^\varphi : \mathbb{R} \curvearrowright M$  is weakly mixing.

Let  $(M, \varphi) = \star_{i=1}^m (M_i, \varphi_i)$  be a non-trivial free product of  $\sigma$ -finite von Neumann algebras endowed with faithful normal states. Let  $E_{M_i} : M \rightarrow M_i$  be the unique  $\varphi$ -preserving conditional expectation.

We first introduce a normal bounded projection onto the subspace of length not greater than 1 as follows. Set

$$\Phi(x) := \varphi(x)1 + \sum_{i=1}^m E_{M_i}(x - \varphi(x)1), \quad x \in M.$$

Clearly, this is a normal self-adjoint linear map from  $M$  to itself. Moreover, we have the following properties:

**Lemma 5.** *The following hold true:*

- (1)  $\Phi \circ \Phi = \Phi$  and  $\Phi(M) = M_1 + \cdots + M_m$ .
- (2)  $\Phi \circ \sigma_t^\varphi = \sigma_t^\varphi \circ \Phi$  for every  $t \in \mathbb{R}$ .
- (3)  $\Phi^\omega : (x_n)^\omega \in M^\omega \mapsto (\Phi(x_n))^\omega \in M^\omega$  defines a well-defined normal self-adjoint idempotent map and its range is exactly  $M_1^\omega + \cdots + M_m^\omega$ .

*Proof.* (1) is just a computation by using  $E_{M_i}(\text{Ker}(\varphi)) = \text{Ker}(\varphi_i)$  and  $E_{M_j}(\text{Ker}(\varphi_i)) = \{0\}$  as long as  $i \neq j$ .

(2) follows from the well-known fact that  $\varphi \circ \sigma_t^\varphi = \varphi$  and  $E_{M_i} \circ \sigma_t^\varphi = \sigma_t^\varphi \circ E_{M_i}$  for  $t \in \mathbb{R}$ .

(3) follows from that the normal self-adjoint map

$$\Psi : x \in M^\omega \mapsto \varphi^\omega(x)1 + \sum_{i=1}^m E_{M_i^\omega}(x - \varphi^\omega(x)1) \in M^\omega$$

satisfies that  $\Psi((x_n)^\omega) = (\Phi(x_n))^\omega = \Phi^\omega((x_n)^\omega)$  for every  $(x_n)^\omega \in M^\omega$  by definition. Here  $E_{M_i^\omega} : M^\omega \rightarrow M_i^\omega$  denotes the  $\varphi^\omega$ -preserving conditional expectation.  $\square$

We will provide three *general* lemmas (Lemmas 6–8), where we always assume that for every  $1 \leq i \leq m$  there exists a  $\varphi_i$ -preserving weak mixing action  $\alpha^{(i)} : G \curvearrowright M_i$  of a common second countable, locally compact group  $G$  and let  $\alpha = \star_{i=1}^m \alpha^{(i)} : G \curvearrowright M$  be the so-called free product action.

The next lemma will be proved by combining the ideas of [9, Theorem 3.1] (whose original idea dates back to Popa's type  $\text{II}_1$  work [17, subsection 2.1]; also see [12, Theorem 3.1] for its finalized version) and [11, Theorem 4.3] (also see [5]). The new essential input here is the observation that the orthogonal projection from  $L^2(M)$  onto  $\overline{L^2(M_1) + \cdots + L^2(M_m)} = \mathbb{C}\xi_\varphi \oplus L^2(M_1)^\circ \oplus \cdots \oplus L^2(M_m)^\circ$  is an extension of  $x\xi_\varphi \in M\xi_\varphi \mapsto \Phi(x)\xi_\varphi = \varphi(x)\xi_\varphi + \sum_{i=1}^m E_{M_i}(x - \varphi(x)1)\xi_\varphi \in M\xi_\varphi$  with the normal self-adjoint linear map  $\Phi : M \rightarrow M$  introduced above, where  $L^2(M_i)^\circ := L^2(M_i) \ominus \mathbb{C}\xi_{\varphi_i}$ ,  $1 \leq i \leq m$ .

**Lemma 6.** *For any  $x \in (M^\omega)^{(\alpha^\omega, G)}$  and  $y, z \in \text{Ker}(\Phi)$  the vectors*

$$y(x - \Phi^\omega(x))\xi_{\varphi^\omega}, \quad (y\Phi^\omega(x) - \Phi^\omega(x)z)\xi_{\varphi^\omega}, \quad (\Phi^\omega(x) - x)z\xi_{\varphi^\omega}$$

*are mutually orthogonal in  $L^2(M^\omega)$ .*

Note that we do not assume that the action  $\alpha^\omega : G \curvearrowright M^\omega$  is continuous in the  $u$ -topology, and simply set  $(M^\omega)^{(\alpha^\omega, G)} := \bigcap_{g \in G} (M^\omega)^{\alpha_g^\omega}$ , being a von Neumann subalgebra of  $M^\omega$ .

*Proof.* Thanks to the Kaplansky density theorem, one can choose bounded nets  $y_\lambda$  and  $z_\lambda$  of elements in the unital  $*$ -subalgebra (algebraically) generated by  $\sigma^{\varphi_i}$ -analytic elements in  $M_i$ ,  $1 \leq i \leq m$ , such that  $\lim_\lambda y_\lambda = y$  and  $\lim_\lambda z_\lambda = z$   $\sigma$ -strongly. Then the  $y_\lambda - \Phi(y_\lambda)$  and the  $z_\lambda - \Phi(z_\lambda)$  fall into  $\text{Ker}(\Phi)$  by Lemma 5(1) and are  $\sigma^\varphi$ -analytic by Lemma 5(2). Moreover,  $\lim_\lambda (y_\lambda - \Phi(y_\lambda)) = y - \Phi(y) = y$  and  $\lim_\lambda (z_\lambda - \Phi(z_\lambda)) = z - \Phi(z) = z$   $\sigma$ -strongly, and furthermore we observe that the  $y_\lambda - \Phi(y_\lambda)$  and the  $z_\lambda - \Phi(z_\lambda)$  fall into the linear span of the reduced words of length  $\geq 2$  whose letters are all  $\sigma^{\varphi_i}$ -analytic. Hence, we may assume that

$y = \sum_{k=1}^{\ell} y_k$  and  $z = \sum_{k=1}^{\ell'} z_k$  such that all  $y_k$  and  $z_k$  are reduced words in the  $M_i^{\circ} := \text{Ker}(\varphi_i)$  of length  $\geq 2$  whose letters are all  $\sigma^{\varphi_i}$ -analytic. By definition we can also write  $x = (x_n)^{\omega}$  with a bounded sequence  $(x_n)$  of elements in  $M$ . Lemma 5(3) shows that  $x - \Phi^{\omega}(x) = (x_n - \Phi(x_n))^{\omega}$  holds for every  $n$ .

For each  $1 \leq i \leq m$  we define the finite dimensional subspace  $V_i$  of  $M_i^{\circ}$  generated by all the  $a, a^*, \sigma_1^{\varphi_i}(a)^*$  of  $M_i^{\circ}$ -letters  $a$  appearing in the  $y_k$  and the  $z_k$ , and denote by  $W_i$  the orthocomplement of  $V_i$  in  $M_i^{\circ}$ , that is, the range of the mapping  $x \in M_i^{\circ} \mapsto x - \sum_{j=1}^{d_i} \varphi_i(v_{ij}^* x) v_{ij} \in M_i^{\circ}$  with an orthogonal basis  $\{v_{ij}\}_{j=1}^{d_i}$  of  $V_i$  with respect to the inner product  $(a|b)_{\varphi_i} := \varphi_i(b^* a)$ ,  $a, b \in M_i$ . Observe that  $M_i^{\circ} = V_i + W_i$  for  $1 \leq i \leq m$ .

Consider the following subspaces  $\mathcal{L}_i$ ,  $\mathcal{R}_i$ ,  $1 \leq i \leq m$ , and  $\mathcal{X}$  of  $L^2(M)$  as follows.

$$\begin{aligned}\mathcal{L}_i &:= [\{\text{reduced words starting in } V_i \text{ of length } \geq 2\} \xi_{\varphi}], \\ \mathcal{R}_i &:= [\{\text{reduced words ending in } V_i \text{ of length } \geq 2\} \xi_{\varphi}], \\ \mathcal{X} &:= [\{\text{reduced words starting and ending in } W_i \text{ (} 1 \leq i \leq m \text{) of length } \geq 2\} \xi_{\varphi}],\end{aligned}$$

where  $[\dots]$  means the operation of closed linear span. Observe that

$$L^2(M) = \mathbb{C} \xi_{\varphi} \oplus L^2(M_1)^{\circ} \oplus \dots \oplus L^2(M_m)^{\circ} \oplus \overline{(\mathcal{L}_1 + \dots + \mathcal{L}_m + \mathcal{R}_1 + \dots + \mathcal{R}_m)} \oplus \mathcal{X}$$

and that the mapping  $a \xi_{\varphi} \mapsto \Phi(a) \xi_{\varphi}$  induces the orthogonal projection from  $L^2(M)$  onto  $\mathbb{C} \xi_{\varphi} \oplus L^2(M_1)^{\circ} \oplus \dots \oplus L^2(M_2)^{\circ}$ .

The first step is to show that  $x \xi_{\varphi}^{\omega} = (P_{\mathcal{X}} x_n \xi_{\varphi} + \Phi(x_n) \xi_{\varphi})^{\omega}$  holds inside  $L^2(M)^{\omega}$ . To this end it suffices to prove that  $\lim_{n \rightarrow \omega} \|P_{\mathcal{L}_i} x_n \xi_{\varphi}\|_{L^2(M)} = \lim_{n \rightarrow \omega} \|P_{\mathcal{R}_i} x_n \xi_{\varphi}\|_{L^2(M)} = 0$  for every  $1 \leq i \leq m$ .

Set

$$\mathcal{H}_L(i) := \sum_{i \neq i(1)}^{\oplus} L^2(M_{i(1)})^{\circ} \oplus \sum_{i \neq i(1) \neq i(2)}^{\oplus} L^2(M_{i(1)})^{\circ} \bar{\otimes} L^2(M_{i(2)})^{\circ} \oplus \dots$$

(the direct sum of all the ‘traveling’ tensor products  $L^2(M_{i(1)})^{\circ} \bar{\otimes} \dots \bar{\otimes} L^2(M_{i(\ell)})^{\circ}$  ( $i(1) \neq i(2) \neq \dots$ ) not starting at  $L^2(M_i)^{\circ}$ ). Then  $\mathcal{L}_i$  may be identified with  $(V_i \xi_{\varphi_i}) \bar{\otimes} \mathcal{H}_L(i) \subseteq L^2(M_i)^{\circ} \bar{\otimes} \mathcal{H}_L(i)$ . Let  $\pi_i : G \curvearrowright L^2(M_i)$  be the canonical implementing unitary representation of the action  $\alpha^{(i)} : G \curvearrowright M_i$ . Similarly, let  $\pi : G \curvearrowright L^2(M)$  be the canonical implementing unitary representation of the action  $\alpha : G \curvearrowright M$ . Then one has  $\pi(g) = \star_{i=1}^m \pi_i(g)$  for  $g \in G$ , that is, its restriction to the reduced subspace  $L^2(M_{i(1)})^{\circ} \bar{\otimes} \dots \bar{\otimes} L^2(M_{i(\ell)})^{\circ}$  is given by  $\pi_{i(1)}(g) \otimes \dots \otimes \pi_{i(\ell)}(g)$ ,  $g \in G$ . Hence the restriction of  $\pi(g)$  to  $\mathcal{L}_i$  is the restriction of  $\pi_i(g) \otimes \pi_{L,i}(g)$  on  $L^2(M_i)^{\circ} \bar{\otimes} \mathcal{H}_L(i)$  to  $(V_i \xi_{\varphi_i}) \bar{\otimes} \mathcal{H}_L(i)$  with a certain unitary representation  $\pi_{L,i} : G \curvearrowright \mathcal{H}_L(i)$ .

For every  $n \in \mathbb{N}$  and every  $g \in G$  we have

$$\begin{aligned}\|P_{\mathcal{L}_i} x_n \xi_{\varphi}\|_{L^2(M)}^2 &= \|\pi(g) P_{\mathcal{L}_i} x_n \xi_{\varphi}\|_{L^2(M)}^2 \\ &= \|\pi(g) P_{\mathcal{L}_i} x_n \xi_{\varphi} - P_{\pi(g)\mathcal{L}_i} x_n \xi_{\varphi} + P_{\pi(g)\mathcal{L}_i} x_n \xi_{\varphi}\|_{L^2(M)}^2 \\ &\leq 2\|P_{\pi(g)\mathcal{L}_i} (\alpha_g(x_n) - x_n) \xi_{\varphi}\|_{L^2(M)}^2 + 2\|P_{\pi(g)\mathcal{L}_i} x_n \xi_{\varphi}\|_{L^2(M)}^2 \\ &\leq 2\|(\alpha_g(x_n) - x_n) \xi_{\varphi}\|_{L^2(M)}^2 + 2\|P_{\pi(g)\mathcal{L}_i} x_n \xi_{\varphi}\|_{L^2(M)}^2.\end{aligned}\tag{1}$$

Let  $K \in \mathbb{N}$  be arbitrarily chosen. According to [8, Proposition 2.3] we choose  $0 < \varepsilon < 1/2$  in such a way that  $\prod_{k=1}^{K-1} (1 + \delta^{\circ k}(\varepsilon))^2 \leq 2$ , where  $\delta^{\circ k}$  means the  $k$ -times composition of the function

$$\delta : t \in [0, 1/2) \mapsto \frac{2t}{\sqrt{1-t} - \sqrt{2t}\sqrt{1-t}} \in [0, +\infty).$$



Let  $\{v_{ij}\}_{j=1}^{d_i}$  be an orthonormal basis of  $V_i$  as before. Then  $\{v_{ij}\xi_{\varphi_i}\}_{j=1}^{d_i}$  becomes an orthonormal basis of the (closed) subspace  $V_i\xi_{\varphi_i}$  of  $L^2(M_i)^\circ$ . By using the weak mixing property of  $\alpha^{(i)} : G \curvearrowright M_i$  one can inductively find a distinct  $2^K$ -tuple  $g_1, \dots, g_{2^K} \in G$  in such a way that

$$\max_{1 \leq j_1, j_2 \leq d_i} |(\pi_i(g_{k_1})v_{ij_1}\xi_{\varphi_i} | \pi_i(g_{k_2})v_{ij_2}\xi_{\varphi_i})_{L^2(M_i)}| \leq \frac{\varepsilon}{d_i} \quad (2)$$

as long as  $k_1 \neq k_2$ . In fact,  $g_1$  can arbitrarily be chosen, and assume that we have chosen  $g_1, \dots, g_p$  in such a way that the above inequality holds for any pair  $k_1 \neq k_2 \leq p$ . Then, applying the weak mixing property to the finite set  $\{\pi_i(g_k)v_{ij} | 1 \leq j \leq d_i, 1 \leq k \leq p\}$  we find  $g_{p+1} \in G$  so that

$$|(\pi_i(g_{p+1})v_{ij_1}\xi_{\varphi_i} | \pi_i(g_k)v_{ij_2}\xi_{\varphi_i})_{L^2(M_i)}| \leq \frac{\varepsilon}{d_i}$$

for all  $1 \leq j_1, j_2 \leq d_i$  and  $1 \leq k \leq p$ . If  $g_{p+1} = g_k$  for some  $1 \leq k \leq p$ , then  $1 = (v_{ij} | v_{ij})_{\varphi_i} = |(\pi_i(g_{p+1})v_{ij_1}\xi_{\varphi_i} | \pi_i(g_k)v_{ij_2}\xi_{\varphi_i})_{L^2(M_i)}| \leq \varepsilon/d_i \leq 1/2$ , a contradiction. Hence  $g_{p+1}$  is different from  $g_1, \dots, g_p$ . In this way, we can inductively obtain the desired family  $g_1, \dots, g_{2^K} \in G$ .

Let  $\xi, \eta \in \mathcal{L}_i = (V_i\xi_{\varphi_i}) \otimes \mathcal{H}_L(i)$  be arbitrarily chosen. Then we can write  $\xi = \sum_{j=1}^{d_i} v_{ij}\xi_{\varphi_i} \otimes \xi_j$  and  $\eta = \sum_{j=1}^{d_i} v_{ij}\xi_{\varphi_i} \otimes \eta_j$  with  $\xi_j, \eta_j \in \mathcal{H}_L(i)$ . If  $k_1 \neq k_2$ , then

$$\begin{aligned} & |(\pi(g_{k_1})\xi | \pi(g_{k_2})\eta)_{L^2(M)}| \\ & \leq \sum_{j_1, j_2=1}^{d_i} |(\pi_i(g_{k_1})v_{ij_1}\xi_{\varphi_i} | \pi_i(g_{k_2})v_{ij_2}\xi_{\varphi_i})_{L^2(M_i)}| \cdot |(\pi_{L,i}(g_{k_1})\xi_{j_1} | \pi_{L,i}(g_{k_2})\eta_{j_2})_{\mathcal{H}_L(i)}| \\ & \leq \frac{\varepsilon}{d_i} \sum_{j_1, j_2=1}^{d_i} \|\xi_{j_1}\|_{\mathcal{H}_L(i)} \cdot \|\eta_{j_2}\|_{\mathcal{H}_L(i)} \quad (\text{use the above (2)}) \\ & \leq \varepsilon \sqrt{\sum_{j=1}^{d_i} \|\xi_j\|_{\mathcal{H}_L(i)}^2} \cdot \sqrt{\sum_{j=1}^{d_i} \|\eta_j\|_{\mathcal{H}_L(i)}^2} \quad (\text{by the Cauchy-Schwarz inequality (} d_i\text{-times)}) \\ & = \varepsilon \|\xi\|_{L^2(M)} \cdot \|\eta\|_{L^2(M)}. \end{aligned}$$

This means that the subspaces  $\pi(g_k)\mathcal{L}_i$ ,  $1 \leq k \leq 2^K$ , are mutually  $\varepsilon$ -orthogonal in the sense of [8, Definition 2.1]. Therefore, by [8, Proposition 2.3] we get

$$\sum_{k=1}^{2^K} \|P_{\pi(g_k)\mathcal{L}_i} x_n \xi_\varphi\|_{L^2(M)}^2 \leq 4 \|x_n \xi_\varphi\|_{L^2(M)}^2,$$

and this and inequality (1) imply that

$$2^K \|P_{\mathcal{L}_i} x_n \xi_\varphi\|_{L^2(M)}^2 \leq 2 \sum_{k=1}^{2^K} \|(\alpha_{g_k}(x_n) - x_n)\xi_\varphi\|_{L^2(M)}^2 + 4 \|x_n \xi_\varphi\|_{L^2(M)}^2.$$

Since  $\alpha_g^\omega(x) = x$  for all  $g \in G$ , we obtain that  $\lim_{n \rightarrow \omega} \|P_{\mathcal{L}_i} x_n \xi_\varphi\|_{L^2(M)}^2 \leq 2^{2-K} \|x \xi_\varphi^\omega\|_{L^2(M^\omega)}^2$ . Since  $K$  can arbitrarily be large, we conclude that  $\lim_{n \rightarrow \omega} \|P_{\mathcal{L}_i} x_n \xi_\varphi\|_{L^2(M)} = 0$ .

In the same way, we also obtain that  $\lim_{n \rightarrow \omega} \|P_{\mathcal{R}_i} x_n \xi_\varphi\|_{L^2(M)} = 0$ . Therefore, we have finished the first step.

The second step is to show that  $y\mathcal{X}$ ,  $y(M_1 + \dots + M_m)\xi_\varphi + (M_1 + \dots + M_m)z\xi_\varphi$  and  $J^M \sigma_{-i/2}^\varphi(z^*)J^M \mathcal{X}$  are mutually orthogonal in  $L^2(M)$ . Since the adjoint  $a^*$  and  $\sigma_1^{\varphi_i}(a)^*$  of any  $M_i^\circ$ -letter  $a$  in the  $y_k$  and  $z_k$  fall in  $V_i$ , it is plain to see that  $y\mathcal{X}$  and  $J^M \sigma_{-i/2}^\varphi(z^*)J^M \mathcal{X}$  sit in

$$[\{\text{reduced words starting in some } V_i \text{ and ending in some } W_i \text{ of length } \geq 3\}\xi_\varphi],$$

$[\{\text{reduced words starting in some } W_i \text{ and ending in some } V_i \text{ of length } \geq 3\}\xi_\varphi],$

respectively. The choice of the subspaces  $V_i, W_i$  guarantees that  $y\mathcal{X}$  and  $J^M\sigma_{-i/2}^\varphi(z^*)J^M\mathcal{X}$  are orthogonal. Since  $y_k^*y_{k'}$  is a linear combination of 1 and reduced words in the  $V_i$ , any element in  $y_k^*y_{k'}M_i$  is a linear combination of an element in  $M_i^\circ$  and reduced words starting in  $V_i$ , and hence  $\mathcal{X}$  and  $y_k^*y_{k'}M_i\xi_\varphi$  are orthogonal, so that so are  $y_k\mathcal{X}$  and  $y_{k'}M_i\xi_\varphi$ . Since any element of  $M_iz_k$  is a linear combination of an element in  $M_i^\circ$  and reduced words ending in some  $V_i$ ,  $y\mathcal{X}$  and  $M_iz_k\xi_\varphi$  are orthogonal. Therefore, we have confirmed that  $y\mathcal{X}$  and  $y(M_1 + \cdots + M_m)\xi_\varphi + (M_1 + \cdots + M_m)z\xi_\varphi$  are orthogonal. In the same way, we can confirm that  $y(M_1 + \cdots + M_m)\xi_\varphi + (M_1 + \cdots + M_m)z\xi_\varphi$  and  $J^M\sigma_{-i/2}^\varphi(z^*)J^M\mathcal{X}$  are orthogonal. Hence we have finished the second step.

Let us finalize this proof. By the first step we have

$$\begin{aligned} y(x - \Phi^\omega(x))\xi_{\varphi^\omega} &= (y(x_n - \Phi(x_n))\xi_\varphi)^\omega = (yP_{\mathcal{X}}x_n\xi_\varphi)^\omega, \\ (y\Phi^\omega(x) - \Phi^\omega(x)z)\xi_{\varphi^\omega} &= ((y\Phi(x_n) - \Phi(x_n)z)\xi_\varphi)^\omega, \\ (\Phi^\omega(x) - x)z\xi_{\varphi^\omega} &= (-J^M\sigma_{-i/2}^\varphi(z^*)J^M(x_n - \Phi(x_n))\xi_\varphi)^\omega \\ &= (-J^M\sigma_{-i/2}^\varphi(z^*)J^MP_{\mathcal{X}}x_n\xi_\varphi)^\omega \end{aligned}$$

in  $L^2(M)^\omega$ . The second step shows that for every  $n \in \mathbb{N}$  the vectors  $yP_{\mathcal{X}}x_n\xi_\varphi$ ,  $(y\Phi(x_n) - \Phi(x_n)z)\xi_\varphi$  and  $J^M\sigma_{-i/2}^\varphi(z^*)J^MP_{\mathcal{X}}x_n\xi_\varphi$  are mutually orthogonal. Thus, the desired assertion immediately follows.  $\square$

The next lemma is a kind of rigidity result of free products of amenable von Neumann algebras, and enough to show that the rigidity of the number of free components in our free product pair rigidity result.

**Lemma 7.** *Any amenable von Neumann subalgebra  $Q$  of  $M$  with separable predual such that  $Q' \cap (M^\omega)^{(\alpha^\omega, G)}$  is diffuse and with expectation must embed into  $M_i$  inside  $M$  for some  $1 \leq i \leq m$ .*

*Proof.* Suppose on the contrary that  $Q \not\leq_M M_i$  for all  $1 \leq i \leq m$ . Proposition 3 together with a direct sum trick (see the proof of [14, Theorem 4.3]) enables us to find a single net  $(u_j)_{j \in J}$  of unitaries in  $Q$  in such a way that  $\lim_j \varphi(u_j) = 0$  as well as  $\lim_j E_{M_i}(u_j) = 0$  strongly for every  $1 \leq i \leq m$ . In particular,  $\lim_j \Phi(u_j) = 0$  strongly. Let  $x \in Q' \cap (M^\omega)^{(\alpha^\omega, G)}$  be arbitrarily chosen. By Lemma 6 we have

$$\begin{aligned} \|(1 - u_j^*\Phi(u_j))(x - \Phi^\omega(x))\xi_{\varphi^\omega}\|_{L^2(M^\omega)} &= \|(u_j - \Phi(u_j))(x - \Phi^\omega(x))\xi_{\varphi^\omega}\|_{L^2(M^\omega)} \\ &\leq \|[x, u_j - \Phi(u_j)]\xi_{\varphi^\omega}\|_{L^2(M^\omega)} \\ &= \|\Phi(u_j), x\|_{L^2(M^\omega)}, \end{aligned}$$

and taking the limit of this inequality along  $j \in J$  we conclude that  $x = \Phi^\omega(x) \in M_1^\omega + \cdots + M_m^\omega$ . Hence we get  $P := Q' \cap (M^\omega)^{(\alpha^\omega, G)} \subseteq M_1^\omega + \cdots + M_m^\omega$ . Here is a claim.

**Claim:**  $P \subseteq M_{i_0}^\omega$  for some  $1 \leq i_0 \leq m$ .

( $\because$ ) Let  $x \in P$  be arbitrarily chosen. Since  $P \subseteq M_1^\omega + \cdots + M_m^\omega = \mathbb{C}1 + (M_1^\omega)^\circ + \cdots + (M_m^\omega)^\circ$ , we write  $x = \alpha 1 + \sum_{i=1}^m x_i^\circ$  with  $x_i^\circ \in (M_i^\omega)^\circ := \text{Ker}(\varphi_i^\omega)$ . This decomposition is uniquely determined thanks to the free independence among the  $M_i^\omega$  (see [23, Proposition 4]). Then we



have

$$\underbrace{(\alpha^2 + \sum_{i=1}^m \varphi^\omega((x_i^\circ)^2))1}_{\in \mathbb{C}1} + \underbrace{\sum_{i=1}^m (2\alpha x_i^\circ + ((x_i^\circ)^2 - \varphi^\omega((x_i^\circ)^2)1))}_{\in M_1^\circ + \dots + M_m^\circ} + \sum_{1 \leq i \neq j \leq m} \underbrace{x_i^\circ x_j^\circ}_{\in M_i^\circ M_j^\circ} = x^2$$

falls into  $P \subseteq \mathbb{C}1 + (M_1^\omega)^\circ + \dots + (M_m^\omega)^\circ$ . Since the  $M_i^\omega$  are freely independent, we observe that if  $i \neq j$ , then  $x_i^\circ x_j^\circ$  must be 0. Hence there exists no pair  $i \neq j$  such that both  $x_i^\circ \neq 0$  and  $x_j^\circ \neq 0$  hold. It follows that each  $x \in P$  falls into  $M_i^\omega$  for some  $1 \leq i \leq m$ .

Choose  $x \in P \setminus \mathbb{C}1$ . (Note that  $P$  is diffuse by assumption.) As shown above there exists  $1 \leq i_0 \leq m$  so that  $x \in M_{i_0}^\omega$ . Suppose that there exist  $1 \leq j \leq m$  with  $j \neq i_0$  and  $y \in P$  such that  $y \in M_j^\omega \setminus M_{i_0}^\omega = M_j^\omega \setminus \mathbb{C}1$ . Then we can write  $x = \alpha 1 + x^\circ$  and  $y = \beta 1 + y^\circ$  with  $x^\circ \in (M_{i_0}^\omega)^\circ$  and  $y^\circ \in (M_j^\omega)^\circ$ . By the choice of  $x$  and  $y$ , we observe that  $x^\circ \neq 0$  and  $y^\circ \neq 0$ . But, as above, we have  $\alpha\beta 1 + (\alpha y^\circ + \beta x^\circ) + x^\circ y^\circ \in M_1^\omega + \dots + M_m^\omega$ , implying that either  $x^\circ = 0$  or  $y^\circ = 0$ , a contradiction. Therefore, all the other  $y \in P$  fall into the  $M_{i_0}^\omega$  containing the element  $x$  that we initially chose. We have proved the claim.

Consider the von Neumann subalgebra  $\bigvee_{i=1}^m M_i^\omega$  of  $M^\omega$  generated by the  $M_i^\omega$ , and observe that  $(\bigvee_{i=1}^m M_i^\omega, \varphi^\omega \upharpoonright_{\bigvee_{i=1}^m M_i^\omega})$  is nothing but the free product of the  $(M_i^\omega, \varphi_i^\omega)$ . Thus, working inside  $M^\omega$  we have

$$Q \subset P' \cap M = P' \cap \left( \bigvee_{i=1}^m M_i \right) \subseteq P' \cap \left( \bigvee_{i=1}^m M_i^\omega \right) \subseteq M_{i_0}^\omega$$

by [12, Proposition 2.7(i)] (see also [24, Proposition 3.1]), since  $P \subseteq M_{i_0}^\omega$  is diffuse and with expectation by assumption. Since the restriction of  $E_{M_{i_0}^\omega}$  to  $M$  is  $E_{M_{i_0}}$ , we have

$$Q \subseteq M_{i_0}^\omega \cap M = E_{M_{i_0}^\omega}(M_{i_0}^\omega \cap M) \subseteq E_{M_{i_0}^\omega}(M) = E_{M_{i_0}}(M) = M_{i_0},$$

a contradiction to  $Q \not\subseteq_M M_{i_0}$ .  $\square$

One may think that the same assertion holds even when  $Q$  is not amenable but has sufficiently many central sequences (i.e, non-full). However this case was already treated as case (ii) in the proof of [13, Main Theorem] without using any group actions. The proof there is rather different from the present one.

The next lemma is a key observation for the rigidity of given states in our free product pair rigidity result.

**Lemma 8.** *Let  $u \in M$  be a unitary and  $1 \leq i \leq m$  be arbitrarily given. Then, if  $M_i$  is diffuse and if  $uM_i u^*$  is globally invariant under the action  $\alpha$ , then  $u$  must fall into  $M_i$ , and hence  $uM_i u^* = M_i$ .*

*Proof.* We use the same notation as in the proof of Lemma 6.

By assumption  $\alpha_g(u)M_i \alpha_g(u)^* = uM_i u^*$  and hence  $M_i = (\alpha_g(u)^* u)M_i (\alpha_g(u)^* u)^*$  for all  $g \in G$ . By [12, Proposition 2.7(i)] we have  $v_g := (\alpha_g(u)^* u) \in M_i$  for every  $g \in G$ . Observe that  $\alpha_g \circ E_{M_i} = E_{M_i} \circ \alpha_g$  holds for every  $g \in G$ . Letting  $x := u^* - E_{M_i}(u^*) \in \text{Ker}(E_{M_i})$  we obtain that  $\alpha_g(x) = v_g x$  for all  $g \in G$ . It suffices to prove that  $x = 0$ .

Observe that  $L^2(M) \ominus L^2(M_i)$  is decomposed into the direct sum of subspaces of the form

$$\mathfrak{X} := [M_i M_{i(1)}^\circ \cdots M_{i(\ell)}^\circ M_i \xi_\varphi] = L^2(M_i) \bar{\otimes} L^2(M_{i(1)})^\circ \bar{\otimes} \cdots \bar{\otimes} L^2(M_{i(\ell)})^\circ \bar{\otimes} L^2(M_i)$$

with  $\ell \geq 1$  and  $i \neq i(1) \neq i(2) \neq \cdots \neq i(\ell) \neq i$ , and thus it suffices to prove that  $P_{\mathfrak{X}} x \xi_\varphi = 0$  for any such  $\mathfrak{X}$ , where  $P_{\mathfrak{X}}$  denotes the projection onto  $\mathfrak{X}$  in  $L^2(M)$ . We remark that  $P_{\mathfrak{X}}$  commutes with the  $v_g$  and the  $\pi(g)$ .

Choose an arbitrary  $\varepsilon > 0$ . We can find a sum  $y = \sum_{k=1}^K w_k$  of words  $w_k$  in  $M_i M_{i(1)}^\circ \cdots M_{i(\ell)}^\circ M_i$  such that  $(2\|x\| + \|y\xi_\varphi - P_{\mathfrak{X}}x\xi_\varphi\|_{L^2(M)})\|y\xi_\varphi - P_{\mathfrak{X}}x\xi_\varphi\|_{L^2(M)} \leq \varepsilon/2$ . Since

$$\pi(g)P_{\mathfrak{X}}x\xi_\varphi = P_{\mathfrak{X}}\pi(g)x\xi_\varphi = P_{\mathfrak{X}}\alpha_g(x)\xi_\varphi = P_{\mathfrak{X}}v_gx\xi_\varphi = v_gP_{\mathfrak{X}}x\xi_\varphi,$$

we have

$$\begin{aligned} & |(\pi(g)y\xi_\varphi|v_gy\xi_\varphi)_{L^2(M)} - \|P_{\mathfrak{X}}x\xi_\varphi\|_{L^2(M)}^2| \\ &= |(\pi(g)y\xi_\varphi|v_gy\xi_\varphi)_{L^2(M)} - (\pi(g)P_{\mathfrak{X}}x\xi_\varphi|v_gP_{\mathfrak{X}}x\xi_\varphi)_{L^2(M)}| \\ &\leq |(\pi(g)(y\xi_\varphi - P_{\mathfrak{X}}x\xi_\varphi)|v_gy\xi_\varphi)_{L^2(M)}| + |(\pi(g)P_{\mathfrak{X}}x\xi_\varphi|v_g(y\xi_\varphi - P_{\mathfrak{X}}x\xi_\varphi))_{L^2(M)}| \\ &\leq \|y\xi_\varphi\|_{L^2(M)} \cdot \|y\xi_\varphi - P_{\mathfrak{X}}x\xi_\varphi\|_{L^2(M)} + \|x\| \cdot \|y\xi_\varphi - P_{\mathfrak{X}}x\xi_\varphi\|_{L^2(M)} \\ &\leq (\|P_{\mathfrak{X}}x\xi_\varphi\|_{L^2(M)} + \|y\xi_\varphi - P_{\mathfrak{X}}x\xi_\varphi\|_{L^2(M)})\|y\xi_\varphi - P_{\mathfrak{X}}x\xi_\varphi\|_{L^2(M)} \\ &\quad + \|x\| \cdot \|y\xi_\varphi - P_{\mathfrak{X}}x\xi_\varphi\|_{L^2(M)} \\ &\leq (2\|x\| + \|y\xi_\varphi - P_{\mathfrak{X}}x\xi_\varphi\|_{L^2(M)})\|y\xi_\varphi - P_{\mathfrak{X}}x\xi_\varphi\|_{L^2(M)}, \end{aligned}$$

and hence

$$\begin{aligned} \|P_{\mathfrak{X}}x\xi_\varphi\|_{L^2(M)}^2 &\leq |(\pi(g)y\xi_\varphi|v_gy\xi_\varphi)_{L^2(M)}| + \frac{\varepsilon}{2} \\ &\leq \sum_{k_1, k_2=1}^K |(\pi(g)w_{k_1}\xi_\varphi|v_gw_{k_2}\xi_\varphi)_{L^2(M)}| + \frac{\varepsilon}{2} \end{aligned}$$

for every  $g \in G$ . For each  $1 \leq k \leq K$ , we write  $w_k = a_k w'_k$ , where  $a_k \in M_i$  is the first  $M_i$ -letter in  $w_k$  and  $w'_k$  denotes the remaining word obtained by removing the first  $M_i$ -letter from  $w_k$ . Observe that  $\mathfrak{X}$  is naturally identified with  $L^2(M_i) \bar{\otimes} \mathfrak{Y}$ , where

$$\mathfrak{Y} := [M_{i(1)}^\circ \cdots M_{i(\ell)}^\circ M_i \xi_\varphi] = L^2(M_{i(1)})^\circ \bar{\otimes} \cdots \bar{\otimes} L^2(M_{i(\ell)})^\circ \bar{\otimes} L^2(M_i). \quad (3)$$

This identification intertwines the restriction of  $\pi$  to  $\mathfrak{X}$  with the tensor product representation of  $\pi_i$  and the restriction of  $\pi$  to  $\mathfrak{Y}$ , and sends each  $w_k \xi_\varphi$  to  $a_k \xi_{\varphi_i} \otimes w'_k \xi_\varphi$ . Observe that

$$\begin{aligned} & |(\pi(g)w_{k_1}\xi_\varphi|v_gw_{k_2}\xi_\varphi)_{L^2(M)}| \\ &= |(\pi_i(g)a_{k_1}\xi_{\varphi_i}|v_ga_{k_2}\xi_{\varphi_i})_{L^2(M_i)}| \cdot |(\pi(g)w'_{k_1}\xi_\varphi|w'_{k_2}\xi_\varphi)_{L^2(M)}| \\ &\leq \left(\max_{1 \leq k \leq K} \|a_k\|\right)^2 |(\pi(g)w'_{k_1}\xi_\varphi|w'_{k_2}\xi_\varphi)_{L^2(M)}| \end{aligned}$$

for all  $1 \leq k_1, k_2 \leq K$ . The restriction of  $\pi(g)$  to  $\mathfrak{Y}$  is identified with  $\pi_{i(1)}(g) \otimes \cdots \otimes \pi_{i(\ell)}(g) \otimes \pi_i(g)$  via the tensor product decomposition in (3), and thus  $\pi : G \curvearrowright \mathfrak{Y}$  is weakly mixing. Hence there exists  $g_\varepsilon \in G$  so that

$$\sum_{k_1, k_2=1}^K |(\pi(g_\varepsilon)w_{k_1}\xi_\varphi|v_{g_\varepsilon}w_{k_2}\xi_\varphi)_{L^2(M)}| \leq \frac{\varepsilon}{2}.$$

Consequently, we get  $\|P_{\mathfrak{X}}x\xi_\varphi\|_{L^2(M)}^2 \leq \varepsilon$ . Since  $\varepsilon > 0$  can arbitrarily be small, we conclude that  $P_{\mathfrak{X}}x\xi_\varphi = 0$ . Hence we are done.  $\square$

We are ready to prove Theorem 1.

*Proof of Theorem 1.* For simplicity, we write  $M_i := \lambda_i^M(R_\infty)$  ( $1 \leq i \leq m$ ) and  $N_j := \lambda_j^N(R_\infty)$  ( $1 \leq j \leq n$ ). In what follows we write  $[m] := \{1, 2, \dots, m\}$  and similarly  $[n] := \{1, 2, \dots, n\}$ .

In order to prove the theorem, we assume that there exists a bijective  $*$ -homomorphism  $\pi : M \rightarrow N$  such that  $\psi = \pi_*(\varphi)$ . Hence we have  $\pi \circ \sigma_t^\varphi = \sigma_t^\psi \circ \pi$ ,  $t \in \mathbb{R}$ , by confirming the

so-called modular condition. Observe that  $N_j$  is globally invariant under  $\sigma^\psi$  and hence so is  $\pi^{-1}(N_j)$  under  $\sigma^\varphi$ . Thanks to [1, Theorem 4.1] we have

$$\pi^{-1}(N_j)' \cap (M^\omega)^{(\sigma^{\varphi^\omega}, \mathbb{R})} = (\pi^{-1}(N_j)' \cap M^\omega)^{(\sigma^{\varphi^\omega}, \mathbb{R})} \supseteq (\pi^{-1}(N_j)' \cap \pi^{-1}(N_j)^\omega)^{(\sigma^{\varphi^\omega}, \mathbb{R})},$$

and the rightmost algebra is isomorphic to the asymptotic centralizer of  $R_\infty$  by [1, Proposition 4.35]. Since  $R_\infty$  is an amenable (or hyperfinite) type III<sub>1</sub> factor,  $\pi^{-1}(N_j)' \cap (M^\omega)^{(\sigma^{\varphi^\omega}, \mathbb{R})}$  must be diffuse thanks to e.g., [21, Proposition 1.1]. Moreover, it is rather trivial in this case that  $\pi^{-1}(N_j)' \cap (M^\omega)^{(\sigma^{\varphi^\omega}, \mathbb{R})}$  is with expectation. Hence, Lemma 7 (with  $G = \mathbb{R}$  and  $\alpha^{(i)} = \sigma^{\varphi_i}$ ) shows that there exists a map  $\kappa_N : [n] \rightarrow [m]$  such that  $\pi^{-1}(N_j) \preceq_M M_{\kappa_N(j)}$  for every  $j \in [n]$ . Therefore, for each  $j \in [n]$  there exist  $d_N(j) \in \mathbb{N}$ , a normal  $*$ -homomorphism  $\Psi : \pi^{-1}(N_j) \rightarrow \mathbb{M}_{d_N(j)}(M_{\kappa_N(j)})$  and a non-zero partial isometry  $v_N(j) \in \mathbb{M}_{d_N(j), 1}(M)$  such that  $\Psi(\pi^{-1}(N_j))$  is with expectation (see Remark 4) and that  $v_N(j)x = \Psi(x)v_N(j)$  for every  $x \in \pi^{-1}(N_j)$ . In particular,  $v_N(j)^*v_N(j) = 1$  thanks to  $\pi^{-1}(N_j)' \cap M = \mathbb{C}1$  by [12, Proposition 2.7(i)]. Remark that the normalizer of  $\pi^{-1}(N_j)$  in  $M$  generates  $\pi^{-1}(N_j)$  itself due to [12, Proposition 2.7(i)], and thus [12, Proposition 2.7(ii)] shows that  $v_N(j)v_N(j)^* \in \mathbb{M}_{d_N(j)}(M_{\kappa_N(j)})$  and

$$v_N(j)\pi^{-1}(N_j)v_N(j)^* \subseteq v_N(j)v_N(j)^*\mathbb{M}_{d_N(j)}(M_{\kappa_N(j)})v_N(j)v_N(j)^*.$$

By symmetry, there also exists a map  $\kappa_M : [m] \rightarrow [n]$  such that  $\pi(M_i) \preceq_N N_{\kappa_M(i)}$ , that is,  $M_i \preceq_M \pi^{-1}(N_{\kappa_M(i)})$  for every  $i \in [m]$ . Moreover, for each  $i \in [m]$  there exist  $d_M(i) \in \mathbb{N}$ , a partial isometry  $v_M(i) \in \mathbb{M}_{1, d_M(i)}(M)$  with  $v_M(i)^*v_M(i) = 1$  such that  $v_M(i)v_M(i)^* \in \mathbb{M}_{d_M(i)}(\pi^{-1}(N_{\kappa_M(i)}))$  and

$$v_M(i)M_iv_M(i)^* \subseteq v_M(i)v_M(i)^*\mathbb{M}_{d_M(i)}(\pi^{-1}(N_{\kappa_M(i)}))v_M(i)v_M(i)^*.$$

With these facts we can proceed exactly in the same way of the final part of the proof of [13, Main Theorem]. However, the present situation allows us to give a bit simpler proof. To make this paper self-contained, we do give it here. Since the  $M_i$  and the  $\pi^{-1}(N_j)$  are all type III factors, we can make the  $v_N(j)$  and the  $v_M(i)$  unitaries in  $M$ . Thus, for each  $i \in [m]$  we have

$$v_N(\kappa_M(i))(v_M(i)M_iv_M(i)^*)v_N(\kappa_M(i))^* \subseteq v_N(\kappa_M(i))\pi^{-1}(N_{\kappa_M(i)})v_N(\kappa_M(i))^* \subseteq M_{\kappa_N(\kappa_M(i))}.$$

Hence, by [13, Lemma 2.8] we get  $\kappa_N(\kappa_M(i)) = i$  for all  $i \in [m]$ . Then, for every  $i \in [m]$  we also get

$$v_N(\kappa_M(i))v_M(i)M_iv_M(i)^*v_N(\kappa_M(i))^* \subseteq M_i,$$

which is actually equality because  $v_N(\kappa_M(i))v_M(i)$  must fall into  $M_i$  by [12, Proposition 2.7(i)]. Therefore, we have

$$v_N(\kappa_M(i))v_M(i)M_iv_M(i)^*v_N(\kappa_M(i))^* = v_N(\kappa_M(i))\pi^{-1}(N_{\kappa_M(i)})v_N(\kappa_M(i))^*,$$

implying that

$$v_M(i)M_iv_M(i)^* = \pi^{-1}(N_{\kappa_M(i)})$$

for all  $i \in [m]$ . By symmetry, we also obtain that  $\kappa_M(\kappa_N(j)) = j$  for all  $j \in [n]$ . Therefore,  $m = n$ . Since  $v_M(i)M_iv_M(i)^* = \pi^{-1}(N_{\kappa_M(i)})$  is globally invariant under  $\sigma^\varphi$ , Lemma 8 shows that  $M_i = \pi^{-1}(N_{\kappa_M(i)})$ ; hence we obtain that  $\pi(M_i) = N_{\kappa_M(i)}$  for all  $i \in [m]$ . Therefore,  $\kappa := \kappa_M \in \mathfrak{S}_m$  is the desired permutation and  $\pi_i := (\lambda_{\kappa(i)}^N)^{-1} \circ (\pi \upharpoonright_{M_i}) \circ \lambda_i^M$  is a well-defined  $*$ -automorphism of  $R_\infty$ , and we have

$$\begin{aligned} (\pi_i)_*(\varphi_i) &= \varphi_i \circ \pi_i^{-1} \\ &= \varphi \circ \lambda_i^M \circ (\lambda_i^M)^{-1} \circ (\pi \upharpoonright_{M_i})^{-1} \circ \lambda_{\kappa(i)}^N \\ &= \varphi \circ (\pi^{-1} \upharpoonright_{N_{\kappa(i)}}) \circ \lambda_{\kappa(i)}^N \end{aligned}$$

$$\begin{aligned}
&= ((\varphi \circ \pi^{-1}) \upharpoonright_{N_{\kappa(i)}}) \circ \lambda_{\kappa(i)}^N \\
&= (\psi \upharpoonright_{N_{\kappa(i)}}) \circ \lambda_{\kappa(i)}^N = \psi_{\kappa(i)}.
\end{aligned}$$

Hence we are done.  $\square$

#### 4. CONCLUDING REMARKS

It is known that the unique amenable type II<sub>1</sub> factor  $R$  admits various weakly mixing actions even of the integers  $\mathbb{Z}$ . For every natural number  $m \geq 2$  we consider the tracial free product type II<sub>1</sub> factor  $M := R^{*m}$ , which is known to be isomorphic to the free group factor  $L(\mathbb{F}_m)$  by Dykema [6] using Voiculescu's free probability theory (see e.g., [28]). The proof of Theorem 1 (see also the proof of Proposition 10 below) actually shows the following: *If any irreducible amenable type II<sub>1</sub> subfactor  $Q \subset M$  with  $Q' \cap M^\omega \not\subseteq Q^\omega$  had weakly mixing actions  $\alpha^{(i)} : G \curvearrowright R$ ,  $1 \leq i \leq m$ , of a second countable, locally compact group  $G$  such that  $Q' \cap (M^\omega)^{(\gamma^\omega, G)}$  was diffuse with  $\gamma := \star_{i=1}^m \alpha^{(i)} : G \curvearrowright M$ , then it would follow that  $L(\mathbb{F}_{r_1}) \cong L(\mathbb{F}_{r_2}) \implies r_1 = r_2$  for any integers  $r_1, r_2 \geq 2$ .* Note that the assumption that  $Q' \cap M^\omega \not\subseteq Q^\omega$  above comes from Popa's spectral gap result [20, Lemma 2]. Remark also that this implication needs only Lemma 7. This strategy to the non-isomorphism problem may not work well, but it seems natural (at least to us) to ask the following question:

**Question 9.** *Let  $\gamma : G \curvearrowright M$  be as above. How large is  $Q' \cap (M^\omega)^{(\gamma^\omega, G)}$  in  $M^\omega$  for an (irreducible) amenable type II<sub>1</sub> subfactor  $Q$  of  $M$  provided that  $Q' \cap M^\omega \not\subseteq Q^\omega$ ?*

We are going to discuss these questions elsewhere in future. However, without knowing any solution to the question, we can prove the next proposition by available techniques.

**Proposition 10.** *Assume that  $\alpha : G \curvearrowright R$  is a weakly mixing action of a second countable, locally compact group. Then, any  $*$ -automorphism of  $M = R^{*m}$  that commutes with the  $m$ -fold free product action  $\alpha^{*m} : G \curvearrowright M$  is obtained as the composition of a permutation over the free components and a free product of  $*$ -automorphisms on  $R$  that commute with the given action  $\alpha$ .*

*Proof.* Denote by  $\beta$  such a  $*$ -automorphism of  $M$ , and also by  $M_i$  the  $i$ th free component (isomorphic to  $R$ ). In the same way as in the proof of Theorem 1 we can prove that there exist a permutation  $\kappa \in \mathfrak{S}_m$ , partial isometries  $v_i \in \mathbb{M}_{d(i),1}(M)$  with  $d(i) \in \mathbb{N}$  so that  $v_i^* v_i = 1$  in  $M_{\kappa(i)}$  (a corner of  $\mathbb{M}_{d(i)}(M)$ ),  $v_i v_i^* \in \mathbb{M}_{d(i)}(\beta^{-1}(M_i))$  and  $v_i M_{\kappa(i)} v_i^* = v_i v_i^* \mathbb{M}_{d(i)}(\beta^{-1}(M_i)) v_i v_i^*$  for every  $1 \leq i \leq m$ . Since  $M_{\kappa(i)} \cong \beta^{-1}(M_i)$  is a type II<sub>1</sub> factor, it is plain to select a unitary  $u_i \in M$  in such a way that  $u_i M_{\kappa(i)} u_i^* = \beta^{-1}(M_i)$  holds. Then Lemma 8 implies that  $M_{\kappa(i)} = u_i M_{\kappa(i)} u_i^* = \beta^{-1}(M_i)$  holds, and the desired assertion immediately follows.  $\square$

This result holds under a variety of other assumptions with allowing infinite index sets thanks to [13, Main Theorem]. The details are left to the reader.

#### ACKNOWLEDGEMENT

Part of the preparation of this manuscript was done during my participation in the Hausdorff Trimester Program “Von Neumann Algebras”, and I am grateful for the financial support and the hospitality of the Hausdorff Research Institute for Mathematics in Bonn. This work was supported in part by my previous Grant-in-Aid for Scientific Research (C) 24540214 as well as my on-going Grant-in-Aid for Challenging Exploratory Research 16K13762.

## REFERENCES

- [1] H. Ando and U. Haagerup, Ultraproducts of von Neumann algebras. *J. Funct. Anal.*, **266** (2014), 6842–6913.
- [2] L. Barnett, Free product von Neumann algebras of type III. *Proc. Amer. Math. Soc.*, **123** (1995), 543–553.
- [3] H. Baumgärtel, *Operatoralgebraic Methods in Quantum Field Theory: a series of lectures*. Akad. Verl., Berlin, 1995.
- [4] N.P. Brown and N. Ozawa, *C\*-algebras and Finite-dimensional Approximations*. Graduate Studies in Mathematics, **88**, American Mathematical Society, Providence, RI, 2008.
- [5] E. Boutonnet and C. Houdayer, Structure of modular invariant subalgebras in free Araki–Woods factors. arXiv:1602.01741.
- [6] K. Dykema, Interpolated free group factors. *Pacific J. Math.*, **163** (1994), 123–135.
- [7] U. Haagerup, Operator valued weights in von Neumann algebras I. *J. Funct. Anal.*, **32** (1979), 175–206.
- [8] C. Houdayer, A class of  $\text{II}_1$  factors with an exotic abelian maximal amenable subalgebra. *Trans. Amer. Math. Soc.*, **366** (2014), 3693–3707.
- [9] C. Houdayer, Gamma stability in free product von Neumann algebras. *Comm. Math. Phys.*, **336** (2015), 831–851.
- [10] C. Houdayer and Y. Isono, Unique prime factorization and bicentralizer problem for a class of type III factors. arXiv:1503.01388v2.
- [11] C. Houdayer and S. Raum, Asymptotic structure of free Araki-Woods factors. *Math. Ann.* **363** (2015), 237–267.
- [12] C. Houdayer and Y. Ueda, Asymptotic structure of free product von Neumann algebras. *Math. Proc. Camb. Phil. Soc.*, to appear. arXiv:1503.02460v2.
- [13] C. Houdayer and Y. Ueda, Rigidity of free product von Neumann algebras. *Compos. Math.*, to appear. arXiv:1507.02157v1.
- [14] A. Ioana, J. Peterson and S. Popa, Amalgamated free products of weakly rigid factors and calculation of their symmetry groups. *Acta Math.*, **200** (2008), 85–153.
- [15] M. Izumi, R. Longo and S. Popa, A Galois correspondence for compact groups of automorphisms of von Neumann algebras with a generalization to Kac algebras. *J. Funct. Anal.*, **155** (1998), 25–63.
- [16] R.V. Kadison and J. Ringrose, *Fundamentals of the theory of operator algebras, Vol. II, Advanced theory*. Graduate Studies in Mathematics, **16**, American Mathematical Society, Providence, RI, 1997.
- [17] S. Popa, Maximal injective subalgebras in factors associated with free groups. *Adv. Math.*, **50** (1983), 27–48.
- [18] S. Popa, On a class of type  $\text{II}_1$  factors with Betti numbers invariants. *Ann. Math.*(2), **163** (2006), 809–899.
- [19] S. Popa, Strong rigidity of  $\text{II}_1$  factors arising from malleable actions of w-rigid groups I. *Invent. math.*, **165** (2006), 369–408.
- [20] S. Popa, On Ozawa’s property for free group factors. *IMRN*, **2007** (2007). doi:10.1093/imrn/rnm036.
- [21] M. Takesaki, The structure of the automorphism group of an AFD factor. “Operator algebras, unitary representations, enveloping algebras, and invariant theory (Paris, 1989)”, 19–44, *Progr. Math.*, **92**, Birkhäuser Boston, Boston, MA, 1990.
- [22] M. Takesaki, *Theory of Operator Algebras II*. Encyclopedia of Mathematical Sciences, **125**, Operator Algebras and Non-commutative Geometry, 6, Springer, Berlin, 2003.
- [23] Y. Ueda, Fullness, Connes’  $\chi$ -groups, and ultra-products of amalgamated free products over Cartan subalgebras. *Trans. Amer. Math. Soc.*, **355** (2003), 349–371.
- [24] Y. Ueda, Factoriality, type classification and fullness for free product von Neumann algebras. *Adv. Math.*, **228** (2011), 2647–2671.
- [25] Y. Ueda, On type  $\text{III}_1$  factors arising as free products. *Math. Res. Lett.*, **18** (2011), 909–920.
- [26] Y. Ueda, Some analysis of amalgamated free products of von Neumann algebras in the non-tracial setting. *J. London Math. Soc.*, **88** (2013), 25–48.
- [27] S. Vaes, Rigidity results for Bernoulli actions and their von Neumann algebras (after Sorin Popa). Séminaire Bourbaki, Vol. 2005/2006, *Astérisque*, **311** (2007), Exp. No. 961, viii, 237–294.
- [28] D.V. Voiculescu, K.J. Dykema and A. Nica, *Free Random Variables*. CRM Monographs, **1**, American Math. Soc. 1992.

GRADUATE SCHOOL OF MATHEMATICS, KYUSHU UNIVERSITY, FUKUOKA, 810-8560, JAPAN

E-mail address: ueda@math.kyushu-u.ac.jp